The Product of Projection Operators

By ISRAEL HALPERIN in Kingston (Canada)

In memory of Maurice Audin

1. Let E_i $(i=1, \ldots, r)$ denote the projection operator onto a subspace M_i of Hilbert space and let $E_1 \wedge \ldots \wedge E_r$ denote the projection operator onto $\bigcap_{i=1}^{r} M_i$. We shall prove the following theorem.

Theorem 1. *Let T denote the product $T=E_1 \ldots E_r$. Then T^n converges strongly to $E_1 \wedge \ldots \wedge E_r$ as $n \to \infty$.*

For the case $r=2$ this theorem was discovered by J. VON NEUMANN and a short, elementary proof was given in his lectures on operator theory at Princeton during 1933–34 (see [3, Theorem 13. 7 on page 55] or [4, Lemma 22 on page 475]). The same theorem, for the case $r=2$, was found later by other authors (H. NAKANO [2, page 48], N. WIENER [7, page 101]).

VON NEUMANN's original proof (and also the others) apply to any Hermitian product $T=E_r \ldots E_2 E_1 E_2 \ldots E_r$ but while for $r=2$ it is easy to pass from $T=E_2 E_1 E_2$ to $T=E_1 E_2$, such an extension is not available when $r>2$. Thus a new proof seems to be required for the case $r>2$.

2. We use the following conventions for an arbitrary Banach space B: M, N denote linear subspaces, T, T_i, P bounded linear operators; $M+N$ means $\{x+y : x \in M, y \in N\}$; $[M] \equiv$ closure of M; $M_0(T) \equiv \{x : Tx=0\}$; $M_1(T) \equiv \equiv \{x : Tx=x\} = M_0(1-T)$; $TM \equiv \{Tx : x \in M\}$ $R(T) \equiv TB$; $K(T) \equiv \sup (|||T^n||| : n=1, 2, \ldots)$.

T will be called a contraction if $|||T||| \leqq 1$, idempotent if $T^2=T$. S will always denote a bounded linear operator with bounded linear inverse S^{-1}, $SS^{-1}=S^{-1}S=1$.

$\varphi(t)$ will denote any non-negative and non-decreasing function defined for $0 \leqq t < \infty$, such that $\varphi(t)>0$ for $t>0$ and $\Phi_2 \equiv \sup \left\{ \dfrac{\varphi(2t)}{\varphi(t)} : 0<t<\infty \right\} < \infty$ (examples of such φ are: $\varphi(t)=t^p$, $p \geqq 1$).

For reference, we list possible properties of T:

(*) $\|Tx\| < \|x\|$ whenever $Tx \neq x$;

(**) $(T^n - T^{n+1})z \to 0$ as $n \to \infty$ for all $z \in B$;

(***) $M_1(T) + [R(1-T)] = B$;

(φ) for some k, $0<k<\infty$, and all $x \in B$,

$$\varphi(\|x-Tx\|) \leqq k\bigl(\varphi(\|x\|) - \varphi(\|Tx\|)\bigr).$$

Now we shall give an elementary proof for the following theorem.

Theorem 2. *Suppose each of the operators T_i $(i=1, \ldots, r)$ on a Hilbert space has property (φ) (with the same function φ) and let T denote the product $T_1 \ldots T_r$. Then T^n converges strongly to the projection onto $\bigcap_{i=1}^{r} M_1(T_i)$ as $n \to \infty$.*

Since all projections have property (φ) (with $\varphi(t)=t^2$ and $k=1$), Theorem 2 includes Theorem 1 for all r.

We shall actually prove a theorem for general Banach spaces from which Theorem 2 follows; if only Theorem 2 is wanted, the proof can be shortened in the obvious way.

3. We observe:

(I) *(φ) implies $(*)$ and $(**)$; $(*)$ implies T is a contraction.* In fact, (φ) implies $\varphi(\|x\|)-\varphi(\|Tx\|)>0$ if $\varphi(\|x-Tx\|)>0$, i. e. $\|x\|>\|Tx\|$ if $\|x-Tx\| \neq 0$. We have also

$$\sum_{n=0}^{N} \varphi\big(\|T^n(1-T)z\|\big)=\sum_{n=0}^{N} \varphi\big(\|T^n z-T(T^n z)\|\big) \leqq \sum_{n=0}^{N} k\big(\varphi(\|T^n z\|)-\varphi(\|T^{n+1}z\|)\big)=$$
$$=k\varphi(\|z\|)-k\varphi(\|T^{N+1}z\|) \leqq k\varphi(\|z\|) \quad \text{for all } N;$$

this implies that $\varphi\big(\|T^n(1-T)z\|\big) \to 0$ as $n \to \infty$, hence $\|T^n(1-T)z\| \to 0$, i. e., $(T^n-T^{n+1})z \to 0$ for all $z \in B$. The rest of (I) is obvious.

(II) *If B is a Hilbert space H and STS^{-1} is a contraction then $(***)$ holds; if T is a contraction then $[R(1-T)]$ is the orthogonal complement of $M_1(T)$.*

If T is a contraction, $|||T^*|||=|||T||| \leqq 1$ and as proved in [5] (see also [6, page 408]), $M_1(T)=M_1(T^*)$ (in fact, $Tx=x$ is equivalent in turn to each of: $(Tx|x)=\|x\|^2$, $(x|T^*x)=\|x\|^2$, $T^*x=x$). Also, as is well-known, $(1-T)^*x=0$ is equivalent to: $((1-T)^*x|y)=0$ for all y, i. e., to: $(x|(1-T)y)=0$, i. e. to: $x \perp R(1-T)$. So $M_1(T)=M_1(T^*)=M_0((1-T)^*)$ and is the orthogonal complement of $[R(1-T)]$, as required.

If STS^{-1} is a contraction the preceding argument shows that $M_1(STS^{-1})+[R(1-STS^{-1})]=H$. Hence $S(M_1(T))+[SR(1-T)]=S(M_1(T)+[R(1-T)])=H$ and so $M_1(T)+[R(1-T)]=H$, i. e. $(***)$ holds for T.

(III) *If B is a Hilbert space P is an idempotent contraction if and only if P is a projection and then P must be the projection onto $M_1(P)$.*

It is known that a projection has these properties. On the other hand, if P is an idempotent contraction then $Px=x$ for $x \in M_1(P)$ and $Px=0$ for $x \in R(1-P)$. By (II), $[R(1-P)]$ is the orthogonal complement of $M_1(P)$.

(IV) *Suppose each ST_iS^{-1} $(i=1, \ldots, r)$ has property $(*)$ and $T=T_1 \ldots T_r$. Then STS^{-1} has property $(*)$ and $Tx=x$ if and only if $T_ix=x$ for all i.*

Consider first the case $S=1$. If $T_ix=x$ for all i then $Tx=T_1 \ldots T_rx=x$. On the other hand if $T_ix \neq x$ for some i, let j be the largest such i; then $\|Tx\|=\|T_1 \ldots T_jx\| \leqq \|T_jx\|<\|x\|$.

Hence if $Tx \neq x$ then $\|Tx\| < \|x\|$ so (*) holds for T and $Tx=x$ if and only if $T_i x = x$ for all i.

The same argument applies with ST_iS^{-1} in place of T_i since $T_ix=x$ and $Tx=x$ are respectively equivalent to $ST_iS^{-1}(Sx)=Sx$ and $STS^{-1}(Sx)=Sx$.

(V) *Suppose each ST_iS^{-1} has property (φ) with $k=k_i$ ($i=1, \ldots, r$), and let $T=T_1 \ldots T_r$. Then STS^{-1} has property (φ).*

Since $STS^{-1}=(ST_1S^{-1})\ldots(ST_rS^{-1})$ we may assume $S=1$. Now

$$\begin{aligned}
&\varphi(\|x-T_1T_2x\|) \leqq \varphi(\|x-T_2x\| + \|T_2x-T_1T_2x\|) \leqq \\
&\leqq \varphi\big(2\max(\|x-T_2x\|, \|T_2x-T_1T_2x\|)\big) \leqq \\
&\leqq \Phi_2\big(\varphi(\|x-T_2x\|)+\varphi(\|T_2x-T_1T_2x\|)\big) \leqq \\
&\leqq \Phi_2 \max(k_1, k_2)\big(\varphi(\|x\|)-\varphi(\|T_2x\|)+\varphi(\|T_2x\|)-\varphi(\|T_1T_2x\|)\big) \leqq \\
&\leqq \Phi_2 \max(k_1, k_2)\big(\varphi(\|x\|)-\varphi\|T_1T_2x\|\big),
\end{aligned}$$

so T_1T_2 has property (φ). By induction, $T_1 \ldots T_r$ has property (φ), as required.

(VI) *If $K(T)$ is finite and T has property (**) then $T^ny \to 0$ as $n \to \infty$ for all $y \in [R(1-T)]$.*

$T^ny \to 0$ as $n \to \infty$ for $y \in R(1-T)$, i. e. for $y=(1-T)z$, by (**), hence for all $y \in [R(1-T)]$ since $K(T) < \infty$.

(VII) *If T^n converges strongly to a limit operator P then $K(T)$ is finite, (**) and (***) hold, P is idempotent, $PT=TP=P$, $Px=x$ if and only if $Tx=x$, $Px=0$ if and only if $x \in [R(1-T)]$.*

Suppose $T^n \to P$. Then $(T^n-T^{n+1}) \to (P-P)=0$ so (**) holds. Also $(\lim T^n)T = \lim T^n = T(\lim T^n)$ since T is bounded, so $PT=P=TP$. Then $PP = (\lim T^n)P = \lim(T^nP) = \lim P = P$ so P is idempotent.

Next, $Px=x$ implies $Tx=TPx=Px=x$ and $Tx=x$ implies $Px=\lim T^nx = \lim x = x$.

Next, since T^n is convergent, $K(T)$ is finite by [1, page 80, Théorème 5]. Since (**) holds, (VI) shows that $T^ny \to 0$ as $n \to \infty$ for all $y \in [R(1-T)]$, i. e. $Py=0$ for $y \in [R(1-T)]$. Now for every $y \in B$,

$$\begin{aligned}
y &= Py+(1-P)y = Py+(1-T^n)(1-P)y+T^n(1-P)y = \\
&= Py+(1-T)(1+T+\ldots+T^{n-1})(1-P)y+T^n(1-P)y.
\end{aligned}$$

Since $T^n(1-P)y \to P(1-P)y=0$ as $n \to \infty$ it follows that $y \in M_1(T)+[R(1-T)]$ so (***) holds. Moreover if $Py=0$ then $y \in [R(1-T)]$. Thus $Py=0$ if and only if $y \in [R(1-T)]$. This proves all parts of (VII).

Theorem 3. (i) *T^n converges strongly to a limit P as $n \to \infty$ if and only if $K(T)$ is finite and (**) and (***) hold.*

Then P is idempotent, $PT=TP=P$, $Px=x$ if and only if $Tx=x$, $Px=0$ if and only if $x \in [R(1-T)]$.

(ii) *If* STS^{-1} *has property* (φ) *then* $K(T)$ *is finite and* (**) *holds (if* $S=1$, P *is an idempotent contraction).*

(iii) *In the case of a Hilbert space,* T^n *does converge to a limit if* (φ) *holds for some* STS^{-1} *(if* $S=1$, P *is the projection onto* $M_1(T)$*).*

Proof of Theorem 3. (i). If $K(T)$ is finite and (**) holds then $T^n(x+y)\to x$ as $n\to\infty$, for all $x\in M_1(T)$ and $y\in[R(1-T)]$, by (VI). Hence if (***) also holds, $T^n(z)$ is convergent for all $z\in B$. The rest of (i) is included in (VII).

Proof of (ii). If (φ) holds for STS^{-1} then STS^{-1} is a contraction and $(**)$ holds for STS^{-1}, by (I). It follows easily that $K(T)$ is finite and (**) holds for T. If T itself is a contraction then (from (I)), P must be a contraction.

(iii) follows from (II) and (III).

Corollary (i). *If each* ST_iS^{-1} $(i=1,\ldots,r)$ *has property* (φ) *and* $T=T_1\ldots T_r$, *then* T^n *converges to a limit* P *as* $n\to\infty$ *if and only if* (***) *holds for* T; P *is idempotent (a contraction, if* $S=1$*) and* $Px=x$ *if and only if* $T_ix=x$ *for all* i.

(ii) *In the case of a Hilbert space if each* ST_iS^{-1} *has property* (φ) *and* $T=T_1\ldots T_r$, *then as* $n\to\infty$, T^n *converges to an idempotent* P, *and we have* $Px=x$ *if and only if* $T_ix=x$ *for all* i *(if* $S=1$, P *is the projection onto* $\bigcap_{i=1}^{r} M(T_i)$*).*

Proof of (i). STS^{-1} has property (φ) by (V) and $Tx=x$ if and only if $T_ix=x$ for all i, by (IV). (i) now follows from Theorem 3 (ii).

Proof of (ii). Because of (V), (ii) follows from Theorem 3 (iii) and (IV).

Note: Corollary (ii) with $S=1$ is identical with Theorem 2.

Added in proof. FELIX BROWDER has kindly informed me that part of the argument of this paper was used previously by S. KAKUTANI to obtain a weaker form of Theorem 1, namely with weak convergence (see F. E. BROWDER, On Some Approximation Methods for Solutions of the Dirichlet Problem for Linear Elliptic Equations of Arbitrary Order, *Journal Math. and Mech.*, 7 (1958), 69—80 for this result).

References

[1] S. BANACH, *Opérations linéaires* (Warsaw, 1932).

[2] H. NAKANO, *Spectral Theory in the Hilbert Space* (Tokyo, 1953).

[3] JOHN VON NEUMANN, *Functional Operators,* Vol. II. (Annals of Mathematics Studies No. 22), (Princeton, 1950). This is a reprint of mimeographed lecture notes first distributed in 1933.

[4] JOHN VON NEUMANN, On Rings of Operators. Reduction Theory, *Annals of Math.*, **50** (1949), 401—485.

[5] F. RIESZ and B. SZ.-NAGY, Über Kontraktionen des Hilbertschen Raumes, *Acta Sci. Math.*, **10** (1941—1943), 202—205.

[6] F. RIESZ and B. SZ.-NAGY, *Functional Analysis* (New York, 1955).

[7] N. WIENER, On the factorization of matrices, *Commentarii Math.*, **29** (1955), 97—111.

QUEEN'S UNIVERSITY
KINGSTON, CANADA

(Received January 8, 1961)